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A Note on the Operators Arising in Spline Approximation

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The purpose of this communication is to establish three theorems about the convergence of sequences of spline approximations. These theorems have close analogs in the classical theory of uniform approximation by ordinary polynomials. These analogs are:

I. If, for each n, L_n is a linear projection of $C[0,1]$ onto the subspace P_n of polynomials of degree $\leqslant n$ then $\|L_n\| \to \infty$.

II. If, for each n, T_n is the (nonlinear) metric projection of $C[0,1]$ onto P_n then $\|f - T_n f\| \leqslant \omega(f; n^{-1})$ whenever $f \in C[0,1]$. By the *metric projection* of f we mean that element of P_n for which $\|f - T_n f\|$ is a minimum.

III. For each n there exists a linear operator A_n from $C[0,1]$ onto P_n such that $\|f - A_n f\| \leqslant \omega(f; n^{-1})$ whenever $f \in C[0,1]$.

We consider here the simplest case of spline approximation. Let C denote the Banach space of all continuous functions f on $[0,1]$ such that $f(0) = f(1)$. The norm in C is $\|f\| = \max\{|f(x)| : 0 \leqslant x \leqslant 1\}$. Let points be prescribed as follows: $0 = x_0 < \ldots < x_n = 1$. In correspondence with these points there is a subspace $S = S(x_0, \ldots, x_n)$ in C whose elements are the cubic splines having nodes at $x_0, \ldots, x_n$. That is, $s \in S$ if and only if $s'' \in C$ and each restriction of s to one of the subintervals $[x_{i-1}, x_i]$ is a cubic polynomial. The dimension of S is n. For each $f \in C$ there is a unique element Lf in S which interpolates to f at the nodes: $f(x_i) = (Lf)(x_i)$ for $i = 0, \ldots n$. The operator L thus defined is a linear projection of C onto S. Reference [*1*] is perhaps the most convenient source of information about these matters.

Now consider a sequence of such nodal arrays $x_i^{(n)}$ $(i = 0, \ldots, n; n = 1, 2, \ldots)$. There corresponds a sequence of subspaces S_n and a sequence of projections L_n. We ask: Under what conditions on the nodes will it be true that $\sup_n \|L_n\| < \infty$? The ultimate desideratum would be a simple formula for calculating $\|L\|$ in

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terms of the nodes. We have not succeeded in discovering such a formula, and indeed there is no reason to believe that one exists. Instead, we have sought upper and lower bounds on $\|L\|$ which are as close as possible.

In order to judge the accuracy of these estimates of $\|L\|$, we consider the following four test cases:

Test Case 1. All n subintervals are of equal length, $1/n$. We obtain then $1 \leqslant \|L_n\| \leqslant 7/4$. In this case, the best upper bound (independent of n) is $(3\sqrt{3}+1)/4$. (This result will appear elsewhere.)

Test Case 2. There are $n-1$ intervals of length $(n+1)/n^2$ and one interval of length $1/n^2$. We obtain the inequality

$$\frac{(3)^{1/2}}{9}\frac{(n+1)^2}{n+2} - 1 \leqslant \|L_n\| \leqslant \frac{3}{2}\frac{(n+1)^2}{n+2} + 1.$$

In this case $\|L_n\| \to \infty$ as $n \to \infty$. By the Uniform Boundedness Theorem, there must exist an $f \in C$ such that $\|L_n f\|$ is unbounded. In [2], Nord investigates such an example and produces a function f and a point x_0 such that $(L_n f)(x_0) \to +\infty$.

Test Case 3. Let $n = 2k+1$, and let $\frac{1}{2} < \theta < 1$. Determine h by the equation $h + 2\theta h + 2\theta^2 h + \ldots + 2\theta^k h = 1$, and let the division of the interval $[0,1]$ be as follows:

$$\begin{array}{l} \qquad \text{etc}\,|\theta^2 h|\theta h|\ h\ |\theta h|\theta^2 h|\,\text{etc} \\ [\!\text{———————————————}\!] \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad 1 \end{array}$$

Our bounds yield the inequality

$$1 \leqslant \|L_n\| \leqslant 19(2\theta - 1)^{-1}.$$

In this case $\sup_n \|L_n\| < \infty$, in spite of the fact that the ratio of the largest to the smallest subinterval becomes infinite.

Test Case 4. This is the same as Test Case 3, except that $0 < \theta < \frac{1}{2}$. Our smallest upper bound becomes infinite, and our largest lower bound remains finite. Hence we are unable to determine whether $\sup \|L_n\| < \infty$.

The bounds on $\|L\|$ are expressed in terms of the following quantities, which depend only on the spacing of the nodes:

$$h_i = x_i - x_{i-1}$$

$$h = \max_{1 \leq i \leq n} h_i$$

$$p_i = h_i/(h_i + h_{i+1})$$

$$q_i = h_{i+1}/(h_i + h_{i-1})$$

$$m_i = \max\{h_i h_{i+1}^{-1}, h_{i+1} h_i^{-1}\}$$

$$m = \max_{1 \le i \le n} m_i$$

$$\alpha_i = \max\{p_i h_{i+1}^{-1}, q_i h_i^{-1}\}$$

$$\alpha = \max_{1 \le i \le n} \alpha_i$$

$$H_i = \max\{h_i^{-1}, h_{i+1}^{-1}\}$$

$$\Lambda_i = \max\{p_{i-1} h_i^{-1}, |p_i h_{i+1}^{-1} - q_i h_i^{-1}|, q_{i+1} h_{i+1}^{-1}\}$$

$$M_i = \Lambda_i + (\Lambda_{i+1} + \Lambda_{i-1}) + \tfrac{1}{2}(\Lambda_{i+2} + \Lambda_{i-2}) + \tfrac{1}{4}(\Lambda_{i+3} + \Lambda_{i-3}) + \dots$$

$$M = \max_{1 \le i \le n} h_i(M_{i-1} + M_i)$$

$$\beta = \min_{1 \le i \le n} \max\{h_i, h_{i+1}\}.$$

THEOREM 1. *The following bounds apply to* L:

(A) $$\|L\| \leqslant \frac{3}{2}\alpha h + 1$$

(B) $$\|L\| \leqslant \frac{4}{9} M + 1$$

(C) $$\|L\| \geqslant 1$$

(D) $$\|L\| \geqslant \frac{(3)^{1/2}}{36} m - 1$$

(E) $$\|L\| \geqslant \frac{(3)^{1/2}}{9} \alpha\beta - 1.$$

THEOREM 2. *For all* $f \in C$, *dist* $(f, S) \leqslant 18\, \omega(f; h)$.

THEOREM 3. *There is a linear operator* $A: C \to S$ *such that* $\|f - Af\| \leqslant 18\omega(f; h)$ *for all* $f \in C$.

Proof of Inequality (A). Let f be any element of C such that $\|f\| \leqslant 1$, and put $s = Lf$, $\lambda_i = s'(x_i), f_i = f(x_i)$. For each $i = 1, \dots, n$ the following equation is valid [*1*, p. 12]:

$$q_i \lambda_{i-1} + 2\lambda_i + p_i \lambda_{i+1} = 3p_i h_{i+1}^{-1}(f_{i-1} - f_i) + 3q_i h_i^{-1}(f_i - f_{i-1}). \tag{1}$$

Let j be an index such that $\max_i|\lambda_i| = |\lambda_j|$. Then from (1),

$$\begin{aligned} 2|\lambda_j| &\leqslant q_j|\lambda_{j-1}| + p_j|\lambda_{j+1}| + 3|p_j h_{j+1}^{-1} f_{j+1} + (q_j h_j^{-1} - p_j h_{j+1}^{-1}) f_j - q_j h_j^{-1} f_{j-1}| \\ &\leqslant (q_j + p_j)\,|\lambda_j| + 3(p_j h_{j+1}^{-1} + |q_j h_j^{-1} - p_j h_{j+1}^{-1}| + q_j h_j^{-1}) \\ &= |\lambda_j| + 6\alpha_j \\ &\leqslant |\lambda_j| + 6\alpha. \end{aligned}$$

This proves that for all i, $|\lambda_i| \leqslant 6\alpha$. Now on the interval $[x_{i-1}, x_i]$ the spline function is given by the following formula

$$s(x) = f_{i-1} A_i(x) + f_i B_i(x) + \lambda_{i-1} C_i(x) + \lambda_i D_i(x) \tag{2}$$

where

$$\begin{aligned} A_i(x) &= h_i^{-3}(h_i + 2x - 2x_{i-1})(x - x_i)^2 \\ B_i(x) &= h_i^{-3}(h_i - 2x + 2x_i)(x - x_{i-1})^2 \\ C_i(x) &= h_i^{-2}(x - x_{i-1})(x - x_i)^2 \\ D_i(x) &= h_i^{-2}(x - x_i)(x - x_{i-1})^2. \end{aligned}$$

We observe that $A_i \geqslant 0$, $B_i \geqslant 0$, $C_i \geqslant 0$, $D_i \leqslant 0$, $A_i + B_i = 1$, and $C_i - D_i = h_i^{-1}(x_i - x)(x - x_{i-1}) \leqslant \frac{1}{4} h_i$. Thus, since $\|f\| \leqslant 1$ and $|\lambda_i| \leqslant 6\alpha$,

$$|s(x)| \leqslant 1 + \tfrac{3}{2}\alpha h_i \leqslant 1 + \tfrac{3}{2}\alpha h.$$

It follows that $\|Lf\| \leqslant 1 + \frac{3}{2}\alpha h$ whenever $\|f\| \leqslant 1$, and that $\|L\| \leqslant 1 + \frac{3}{2}\alpha h$.

Proof of Inequality (B). For each index $j = 1, \ldots, n$ there is a spline function s^j such that $s^j(x_i) = \delta_i{}^j$ for $i = 1, \ldots, n$. This spline function is termed the "jth cardinal function"; in terms of it, the spline operator L can be expressed in the form $Lf = \sum_{j=1}^n f(x_j) s^j$. From this it follows that $\|L\| = \|g\|$, where $g(x) = \sum_{j=1}^n |s^j(x)|$. We define $\lambda_i{}^j = (s^j)'\,(x_i)$ and $\|\lambda^j\| = \max_{1 \leq i \leq n}|\lambda_i{}^j|$. The numbers $\lambda_1{}^j, \ldots, \lambda_n{}^j$ satisfy the system of equations

$$q_i \lambda_{i-1}^j + 2\lambda_i{}^j + p_i \lambda_{i+1}^j = R_i{}^j \qquad (i = 1, \ldots, n)$$

in which $R_i{}^j = 3p_i h_{i+1}^{-1}(\delta_{i+1}^j - \delta_i{}^j) + 3q_i h_i^{-1}(\delta_i{}^j - \delta_{i-1}^j)$.

Assertion 1. For each $k = 0, 1, 2, \ldots, [n/2]$ the inequality $|\lambda_i{}^j| \leqslant 2^{-k}\|\lambda^j\|$ is valid for pairs (i, j) satisfying $|i - j| > k$. (Computations involving the indices are carried out in arithmetic modulo n because of periodicity.) In order to prove this assertion, we use induction on k. For $k = 0$ the inequality is trivial. If the assertion is true for an index $k \geqslant 0$, then it is true for $k + 1$. Indeed, suppose that $|i - j| > k + 1$. Then $|i + 1 - j| > k$ and $|i - 1 - j| > k$. Also $|i - j| > 1$. Hence $R_i{}^j = 0$. From Eq. (3) we have $2|\lambda_i{}^j| = |q_i \lambda_{i-1}^j + p_i \lambda_{i+1}^j| \leqslant \max\{|\lambda_{i-1}^j|, |\lambda_{i+1}^j|\} \leqslant 2^{-k}\|\lambda^j\|$. Thus $|\lambda_i{}^j| \leqslant 2^{-k-1}\|\lambda^j\|$.

ASSERTION 2. $\|\lambda^j\| \leqslant 3\Lambda_j$. In order to establish this, let k be an index such that $|\lambda_k{}^j| = \|\lambda^j\|$. From Eq. (3), we have $2\|\lambda^j\| = 2|\lambda_k{}^j| = |R_k{}^j - p_k\lambda_{k+1}^j - q_k\lambda_{k-1}^j| \leqslant |R_k{}^j| + (p_k + q_k) \max\{|\lambda_{k+1}^j|, |\lambda_{k-1}^j|\} \leqslant |R_k{}^j| + \|\lambda^j\|$. Thus $\|\lambda^j\| \leqslant \max_i |R_i{}^j|$. Now, all the numbers $R_1{}^j, \ldots, R_n{}^j$ vanish with the exception of these three:

$$|R_{j-1}^j| = 3p_{j-1} h_j^{-1} \leqslant 3\Lambda_j;$$
$$|R_j{}^j| = 3|p_j h_{j+1}^{-1} - q_j h_j^{-1}| \leqslant 3\Lambda_j;$$
$$|R_{j+1}^j| = 3q_{j+1} h_{j+1}^{-1} \leqslant 3\Lambda_j.$$

ASSERTION 3. $\sum_{j=1}^n |\lambda_i{}^j| \leqslant 3M_i$. For the proof, we use Assertions 1 and 2 as follows:

$$\begin{aligned}\sum_{j=1}^n |\lambda_i{}^j| &= |\lambda_i{}^i| + |\lambda_i^{i-1}| + |\lambda_i^{i+1}| + |\lambda_i^{i-2}| + |\lambda_i^{i+2}| + \ldots \\ &\leqslant \|\lambda^i\| + \|\lambda^{i-1}\| + \|\lambda^{i+1}\| + \tfrac{1}{2}\|\lambda^{i-2}\| + \tfrac{1}{2}\|\lambda^{i+2}\| + \ldots \\ &\leqslant 3(\Lambda_i + \Lambda_{i-1} + \Lambda_{i+1} + \tfrac{1}{2}\Lambda_{i-2} + \tfrac{1}{2}\Lambda_{i+2} + \ldots).\end{aligned}$$

Now for the proof of Inequality (B), let x be any point of $[0, 1]$. Let i be an index such that $x_{i-1} \leqslant x \leqslant x_i$. An elementary calculation shows that $0 \leqslant C_i(x) \leqslant (4/27)h_i$ and $0 \leqslant -D_i(x) \leqslant (4/27)h_i$. Thus by Eq. (2) and Assertion 3,

$$\begin{aligned}g(x) &= \sum_{j=1}^n |s^j(x)| \\ &= \sum_{j=1}^n |\delta_{i-1}^j A_i(x) + \delta_i{}^j B_i(x) + \lambda_{i-1}^j C_i(x) + \lambda_i{}^j D_i(x)| \\ &\leqslant A_i(x) + B_i(x) + C_i(x) \sum_{j=1}^n |\lambda_{i-1}^j| - D_i(x) \sum_{j=1}^n |\lambda_i{}^j| \\ &\leqslant 1 + 3(M_{i-1} + M_i) \max\{C_i(x), -D_i(x)\} \\ &\leqslant 1 + 4/9\, h_i(M_{i-1} + M_i) \leqslant 1 + 4/9\, M.\end{aligned}$$

Inequality (C) is trivial since $L1 = 1$.

Proof of Inequality (D). Select an index j such that $m_j = m$. Then either $h_j h_{j+1}^{-1} = m$ or $h_{j+1} h_j^{-1} = m$, and without loss of generality we assume the latter. Consider now the jth cardinal spline function s^j, and the numbers $R_i{}^j$, $\lambda_i{}^j$, $\|\lambda^j\|$ as defined in the proof of Inequality (B). In the following, superscripts j will be omitted for simplicity.

ASSERTION 4. $\|\lambda\| \leqslant 3m(1 + m)^{-1} h_j^{-1}$. In order to prove this, we start with Assertion 2: $\|\lambda\| \leqslant 3\Lambda_j$. From the definition of m, we have $h_i h_{i+1}^{-1} \leqslant m$ and $h_{i+1} h_i^{-1} \leqslant m$ for all i. Since $p_i = h_i/(h_i + h_{i+1}) = 1/(1 + h_{i+1} h_i^{-1})$, we see that

$1/(1+m) \leqslant p_i \leqslant m/(1+m)$. The same inequality is true for all the coefficients q_i. Thus from the definition of Λ_i we have $\Lambda_j \leqslant m(1+m)^{-1} \max \{h_j^{-1}, h_{j+1}^{-1}\} = m(1+m)^{-1} h_j^{-1}$.

ASSERTION 5. $|\lambda_{j-2}| \leqslant \frac{1}{2}\|\lambda\|$. This follows from Assertion 1.

ASSERTION 6. Define the functions $P(m) = 3m^{-3}(m^3 + 2m^2 - 2m - 2)$ and $Q(m) = m^{-2}(4m^2 + 9m + 6)$. Then $h_j|\lambda_{j-2}| \geqslant P(m) - Q(m)\theta$, where $\theta = h_j \max \{|\lambda_j|, |\lambda_{j+1}|\}$. In order to prove this, replace i by j in Eq. (3) and solve (3) for λ_{j-1}. The result is $\lambda_{j-1} = q_j^{-1}(R_j - 2\lambda_j - p_j\lambda_{j+1})$. Now replace i by $j-1$ in Eq. (3) and solve for λ_{j-2}. We obtain

$$h_j|\lambda_{j-2}| \geqslant -h_j\lambda_{j-2} = h_j q_{j-1}^{-1}(-R_{j-1} + 2\lambda_{j-1} + p_{j-1}\lambda_j).$$

In this equation replace λ_{j-1} by its value computed above, express R_{j-1} and R_j by their values, and finally replace λ_j and λ_{j+1} by their upperbound, θh_j^{-1}. The result is

$$\begin{aligned} h_j|\lambda_{j-2}| \geqslant h_j q_{j-1}^{-1}[&2q_j^{-1}(3q_j h_j^{-1} - 3p_j h_{j+1}^{-1}) - 3p_{j-1} h_j^{-1} \\ &- (4q_j^{-1} - p_{j-1})\theta h_j^{-1} - 2q_j^{-1} p_j \theta h_j^{-1}]. \end{aligned}$$

Since $q_j p_j^{-1} = h_{j+1} h_j^{-1} = m$, $p_{j-1} = 1 - q_{j-1}$, $q_j^{-1} = (m+1)m^{-1}$, $p_{j-1} \geqslant (m+1)^{-1}$, and $q_{j-1}^{-1} \geqslant (m+1)m^{-1}$, we obtain

$$\begin{aligned} h_j|\lambda_{j-2}| &\geqslant (m+1)m^{-1}\{3 - 6m^{-2} + 3(m+1)^{-1} \\ &\qquad - [4(m+1)m^{-1} - (m+1)^{-1} + 2m^{-1}]\theta\} \\ &= 3m^{-3}(m^3 + 2m^2 - 2m - 2) - m^{-2}(4m^2 + 9m + 6)\theta. \end{aligned}$$

ASSERTION 7. If $m \geqslant 2$, then $4P(m) - Q(m) > 6m(1+m)^{-1}$. In order to prove this, it is enough to prove that $4m^3(1+m)P(m) - m^3(1+m)Q(m) - 6m^4 > 0$. The expression on the left turns out to be $2m^4 + 23m^3 - 15m^2 - 54m - 24$, and this is positive when $m \geqslant 2$.

ASSERTION 8. If $m \geqslant 2$, then $\max \{|\lambda_j|, |\lambda_{j+1}|\} > (4h_j)^{-1}$. If this inequality is false, then $\theta \leqslant \frac{1}{4}$ and by Assertions 6, 7, 4, 5 we have the following contradiction:

$$\begin{aligned} |\lambda_{j-2}| &\geqslant [P(m) - Q(m)\theta] h_j^{-1} \\ &\geqslant [P(m) - \tfrac{1}{4}Q(m)] h_j^{-1} \\ &> \tfrac{3}{2}m(1+m)^{-1} h_j^{-1} \\ &\geqslant \tfrac{1}{2}\|\lambda\|. \end{aligned}$$

ASSERTION 9. $\|L\| \geqslant (\sqrt{3}/36)m - 1$. In order to establish this, let f denote a function such that $f_j = 1$, $f_i = -1$ when $i \neq j$, and $\|f\| = 1$. Put $g = Lf$. Since

$L1 = 1, g = 2s - 1$. (Here s is the jth cardinal function.) Hence $\|L\| \geqslant \|Lf\| = \|g\|$. On the interval $[x_j, x_{j+1}]$,

$$\begin{aligned} |g(x)| &= |f_j A_{j+1}(x) + f_{j+1} B_{j+1}(x) + g_j' C_{j+1}(x) + g_{j+1}' D_{j+1}(x)| \\ &= |A_{j+1}(x) - B_{j+1}(x) + 2\lambda_j C_{j+1}(x) + 2\lambda_{j+1} D_{j+1}(x)|. \end{aligned}$$

If $|\lambda_j| \geqslant |\lambda_{j+1}|$, then we take $x = x_j + th_{j+1}$ with $t = \frac{1}{2} - \frac{1}{6}\sqrt{3}$, and use Assertion 8 to write

$$\begin{aligned} |g(x)| &\geqslant 2|\lambda_j|\,|C_{j+1}(x)| - 2|\lambda_{j+1}|\,|D_{j+1}(x)| - |A_{j+1}(x) - B_{j+1}(x)| \\ &\geqslant 2|\lambda_j|\,[C_{j+1}(x) + D_{j+1}(x)] - 1 \\ &\geqslant (2h_j)^{-1}(\sqrt{3}\,h_{j+1}/18) - 1 \\ &= (\sqrt{3}/36)m - 1. \end{aligned}$$

On the other hand, if $|\lambda_{j+1}| > |\lambda_j|$, we take $t = \frac{1}{2} + \frac{1}{6}\sqrt{3}$ and write

$$\begin{aligned} |g(x)| &\geqslant 2|\lambda_{j+1}|\,|D_{j+1}(x)| - 2|\lambda_j|\,|C_{j+1}(x)| - 1 \\ &\geqslant (\sqrt{3}/36)m - 1. \end{aligned}$$

Proof of Inequality (E). Let j be an index such that $\alpha_j = \alpha$. Let f be an element of C such that $\|f\| = 1$, $f_j = \operatorname{sgn}(q_j h_j^{-1} - p_j h_{j+1}^{-1})$, $f_{j+1} = -1$, and $f_{j-1} = 1$. The system of Eq. (1) is of the form $A\lambda = b$, where λ and b are n-tuples and A is an $n \times n$ matrix. If the vector norm is $\|\lambda\| = \max|\lambda_i|$, then the matrix norm is $\|A\| = \max_i \sum_j |A_{ij}|$. Hence from the inequality $\|b\| \leqslant \|A\|\,\|\lambda\|$, we obtain

$$\|\lambda\| \geqslant \|b\|/\|A\| \geqslant b_j/\max_i\,(q_i + 2 + p_i) = 2\alpha.$$

Now select an index k such that $|\lambda_k| = \|\lambda\|$. We consider two cases. First, if $h_k \geqslant h_{k+1}$, then $h_k \geqslant \beta$. We examine $s(x)$ on $[x_{k-1}, x_k]$, using Eq. (2). The result is

$$\begin{aligned} |s(x)| &\geqslant |\lambda_k|\,|D_k(x)| - |\lambda_{k-1}|\,|C_k(x)| - |A_k(x)| - |B_k(x)| \\ &\geqslant |\lambda_k|\,[-D_k(x) - C_k(x)] - 1. \end{aligned}$$

We take $x = x_{k-1} + \theta h_k$ with $\theta = \frac{1}{2} + \frac{1}{6}\sqrt{3}$ and obtain

$$\|L\| \geqslant \|Lf\| = \|s\| \geqslant |s(x)| \geqslant (2\alpha)\,[h_k(\sqrt{3}/18)] - 1 \geqslant (\sqrt{3}/9)\,\alpha\beta - 1.$$

In the second case, $h_{k+1} \geqslant h_k$, so that $h_{k+1} \geqslant \beta$. Examining $s(x)$ on the interval $[x_k, x_{k+1}]$, we obtain the bound

$$\begin{aligned} |s(x)| &\geqslant |\lambda_k|\,|C_{k+1}(x)| - |\lambda_{k+1}|\,|D_{k+1}(x)| - |A_{k+1}(x)| \\ &\qquad - |B_{k+1}(x)| \\ &\geqslant |\lambda_k|\,[C_{k+1}(x) + D_{k+1}(x)] - 1. \end{aligned}$$

If $x = x_{k+1} - \theta h_{k+1}$, then as before, $\|L\| \geqslant 2\alpha[h_{k+1}(\sqrt{3}/18)] - 1 \geqslant (\sqrt{3}/9)\alpha\beta - 1$.

Proof of Theorem 2. Given f and $h > 0$, we define

$$g(x) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)\,dt.$$

Since f is continuous, g is continuously differentiable. Furthermore, $g'(x) = (1/h)[f(x+h/2) - f(x-h/2)]$, whence $\|g'\| \leqslant (1/h)\,\omega(f;\ h)$. From the equation

$$g(x) - f(x) = \frac{1}{h}\int_{x-h/2}^{x+h/2} [f(t) - f(x)]\,dt$$

we obtain $\|g - f\| \leqslant \omega(f; h)$. By a Lemma proved below, the spline $s = Lg$ has the property $\|g - s\| \leqslant (17/2)h\omega(g'; h)$. From the obvious inequality $\omega(g'; h) \leqslant 2\|g'\|$ we obtain $\|g - s\| \leqslant 17h\|g'\| \leqslant 17\omega(f; h)$. Thus

$$\begin{aligned} \operatorname{dist}(f, S) &\leqslant \|f - s\| \\ &\leqslant \|f - g\| + \|g - s\| \\ &\leqslant 18\omega(f; h). \end{aligned}$$

Proof of Theorem 3. It is only necessary to observe that the function $s \equiv Lg$ in the preceding proof depends linearly upon f.

The following Lemma, with the constant 76 in place of 17/2, was given by Sharma and Meir in [*3*, p. 763]. Then, with the constant 21/2 it was proved by Ahlberg, Nilson, and Walsh in [*1*, p. 27].

LEMMA. *Let $f' \in C$ and $s = Lf$. Then $\|f' - s'\| \leqslant (17/2)\omega(f'; h)$ and $\|f - s\| \leqslant (17/2)h\omega(f'; h)$.*

Proof. The second inequality is a consequence of the first. See [1, p. 27].

In order to prove the first inequality, we start with the Eq. (1), and use the mean-value theorem to write

$$q_i\lambda_{i-1} + 2\lambda_i + p_i\lambda_{i+1} = 3p_i f'(\xi_i) + 3q_i f'(\xi_{i-1}).$$

For convenience put $a_i = \lambda_i - f'(x_i)$. Then

$$\begin{aligned} q_i a_{i-1} + 2a_i + p_i a_{i+1} &= \text{“R.H.”} \\ &\equiv 2p_i[f'(\xi_i) - f'(x_i)] + 2q_i[f'(\xi_{i-1}) - f'(x_i)] \\ &\quad + p_i[f'(\xi_i) - f'(x_{i+1})] + q_i[f'(\xi_{i-1}) - f'(x_{i-1})] \end{aligned}$$

Suppose that j is the index of the largest $|a_i|$. Then

$$\begin{aligned} 2|a_j| \leqslant q_j|a_{j-1}| + p_j|a_{j+1}| + |\text{R.H.}| &\leqslant q_j|a_j| \\ &\quad + p_j|a_j| + |\text{R.H.}|. \end{aligned}$$

Hence $|a_j| \leqslant |\text{R.H.}| \leqslant 3\omega(f';h)$. From this point on, the proof is the same as in [*1*].

The following questions remain open:

1. What conditions on the nodes are equivalent to the inequality $\sup_n \|L_n\| < \infty$?

2. Is there a linear projection A of C onto S such that $\|f - Af\| \leqslant c\omega(f;h)$?

3. What is the linear projection of minimum norm from C onto S? Is it unique?

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